ON THE THEOREM OF POINCARE CONCERNING A MOTION OF A RIGID BODY IN A NEWTONIAN FORCE FIELD

(OB ODNOI TEOREME PUANKARE, OTNOSIASHCHEISIA K ZADACHE O DVIZHENII TVERDOGO TELA V NIUTONOVSKOM POLE SIL)

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1. It is well known [1] that the approximate equations of motion of a rigid body about a fixed point in a central Newtonian force field

$$\begin{gathered}
A\frac{dp}{dt} + (C-B)\,qr = y'_0\gamma'' - z'_0\gamma' + \alpha\,(C-B)\,\gamma'\gamma'' \\
B\frac{dq}{dt} + (A-C)\,pr = z'_0\gamma - x'_0\gamma'' + \alpha\,(A-C)\,\gamma''\gamma \\
C\frac{dr}{dt} + (B-A)\,pq = x'_0\gamma' - y'_0\gamma + \alpha\,(B-A)\,\gamma\gamma' \qquad (1.1) \\
\frac{d\gamma}{dt} = r\gamma' - q\gamma'', \qquad \frac{d\gamma'}{dt} = p\gamma'' - r\gamma, \qquad \frac{d\gamma''}{dt} = q\gamma - p\gamma' \\
\left(x'_0 = Mgx_0,\ y'_0 = Mgy_0,\ z'_0 = Mgz_0,\ \alpha = \frac{3g}{R}\right)
\end{gathered}$$

have three independent first integrals; the kinetic energy integral, the integral of the areas and the trivial integral

$$\begin{gathered}
Ap^2 + Bq^2 + Cr^2 - 2\,(x'_0\gamma + y'_0\gamma' + z'_0\gamma'') + \alpha\,(A\gamma^2 + B\gamma'^2 + C\gamma''^2) = \text{const} \qquad (1.2) \\
Ap\gamma + Bq\gamma' + Cr\gamma'' = \text{const}, \qquad \gamma^2 + \gamma'^2 + \gamma''^2 = 1
\end{gathered}$$

Since the system (1.1) does not contain the time t explicitely and since the last Jacobi multiplier is unity, we would be able to solve the problem by quadratures if we could obtain four independent first integrals which do not involve the time. The three independent first integrals (1.2) are algebraic. The two cases

$$(1) \quad x_0 = y_0 = z_0 = 0, \qquad (2) \quad A = B,\ x_0 = y_0 = 0$$

are the only cases when the system (1.1) was solved by quadratures. The solutions are single-valued functions [2] of time and all four algebraic first integrals are readily obtained. In the first case (which is analogous to Euler's case in the classical problem of motion of a rigid body about a fixed point in a uniform gravitational field) the fourth integral is

$$Ap^2 + Bq^2 + Cr^2 - \alpha(BC\gamma^2 + AC\gamma'^2 + AB\gamma''^2) = \text{const}$$

In the second case (which is analogous to Lagrange's case in the same classical problem) the fourth integral is

$$r = \text{const}$$

The following question arises: under what other conditions is the existence of a fourth algebraic integral possible?

We shall demonstrate that for the problem which we consider the following theorem of Poincaré [3, 4] is applicable: the necessary condition for the existence of an additional algebraic integral of the system (1.1) is that the ellipsoid of inertia about the fixed point must be an ellipsoid of revolution.

2. Introducing the new variables y_1, y_2, z_1, z_2, instead of p, q, γ, γ' (assuming that A, B, C are distinct)

$$y_1 = \sqrt{A(A-C)}\,p + i\sqrt{B(B-C)}\,q, \qquad z_1 = \gamma + i\gamma'$$
$$y_2 = \sqrt{A(A-C)}\,p - i\sqrt{B(B-C)}\,q, \qquad z_2 = \gamma - i\gamma'$$

and replacing y_1, z_1, z_2, γ'', t by λy_1, λz_1, λz_2, $\lambda\gamma''$, $-it$, where λ is an arbitrary parameter, we transform the system (1.1) into

$$\frac{dy_1}{dt} = -\sqrt{\frac{(A-C)(B-C)}{AB}}\,ry_1 - \left[x'_0\sqrt{\frac{B-C}{B}} + iy'_0\sqrt{\frac{A-C}{A}}\right]\gamma'' + \frac{z'_0}{2}\left[\sqrt{\frac{B-C}{B}}(z_1+z_2) + \sqrt{\frac{A-C}{A}}(z_1-z_2)\right] - \lambda\frac{\alpha\gamma''}{2}\left[(C-B)\sqrt{\frac{A-C}{A}}(z_1-z_2) - (A-C)\sqrt{\frac{B-C}{B}}(z_1+z_2)\right]$$

$$\frac{dy_2}{dt} = \sqrt{\frac{(A-C)(B-C)}{AB}}\,ry_2 + \left[x'_0\sqrt{\frac{B-C}{B}} - iy'_0\sqrt{\frac{A-C}{A}}\right]\lambda\gamma'' - \frac{\lambda z'_0}{2}\left[\sqrt{\frac{B-C}{B}}(z_1+z_2) - \sqrt{\frac{A-C}{A}}(z_1-z_2)\right] - \lambda^2\frac{\alpha\gamma''}{2}\left[(C-B)\sqrt{\frac{A-C}{A}}(z_1-z_2) + (A-C)\sqrt{\frac{B-C}{B}}(z_1+z_2)\right] \qquad (2.1)$$

$$C\frac{dr}{dt} = \frac{A-B}{4\sqrt{AB(A-C)(B-C)}}(y_2^2 - \lambda^2 y_1^2) - \frac{\lambda}{2}\left[x'_0(z_1-z_2) - iy'_0(z_1+z_2)\right] + \frac{\lambda^2\alpha}{4}(A-B)(z_1^2 - z_2^2)$$

$$\frac{dz_1}{dt} = -rz_1 + \frac{1}{2}\left[\frac{\lambda y_1 + y_2}{\sqrt{A(A-C)}} + \frac{\lambda y_1 - y_2}{\sqrt{B(B-C)}}\right]\gamma''$$

$$\frac{dz_2}{dt} = rz_2 - \frac{1}{2}\left[\frac{\lambda y_1 + y_2}{\sqrt{A(A-C)}} - \frac{\lambda y_1 - y_2}{\sqrt{B(B-C)}}\right]\gamma''$$

$$\frac{d\gamma''}{dt} = \frac{1}{4\sqrt{A(A-C)}}(\lambda y_1 + y_2)(z_1 - z_2) - \frac{1}{4\sqrt{B(B-C)}}(\lambda y_1 - y_2)(z_1 + z_2)$$

This system has the following first algebraic integrals:

$$\frac{(\lambda y_1 + y_2)^2}{A-C} - \frac{(\lambda y_1 - y_2)^2}{B-C} + 4Cr^2 - 4\lambda\left[x'_0(z_1+z_2) - iy'_0(z_1-z_2) + 2z'_0\gamma''\right] +$$
$$+ \lambda^2\alpha\left[A(z_1+z_2)^2 - B(z_1-z_2)^2 + 4C\gamma''^2\right] = h_1 \tag{2.2}$$
$$\frac{A}{2\sqrt{A(A-C)}}(\lambda y_1 + y_2)(z_1+z_2) - \frac{B}{2\sqrt{B(B-C)}}(\lambda y_1 - y_2)(z_1 - z_2) + Cr\gamma'' = h_2$$
$$z_1z_2 + \gamma''^2 = h_3$$

where h_1, h_2, h_3 are arbitrary constants.

When Husson [5] proved the theorem of Poincaré for the motion of a heavy solid about a fixed point in a uniform gravitational force field, he used the system of equations (2.1) with $\alpha = 0$, and both in the system (2.1) and in its first integrals λ was set equal to zero. The theorem applies also the the case when $\lambda \neq 0$, because the right members of the differential equations and the first integrals are polynomials of y_1, y_2, z_1, z_2, r, γ'', λ.

Since at $\lambda = 0$ the equations (2.1) and their first integrals are independent of α, and since at $\lambda \neq 0$, $\alpha \neq 0$, the right members of the equations (2.1) and the expressions (2.2) are polynomials in y_1, y_2, z_1, z_2, r, γ'', α, the proof of Husson applies also to our problem.

Thus for the problem of motion of a rigid body about a fixed point under the action of a central Newtonian force field and with arbitrary initial conditions we have the following theorem of Poincaré: if the ellipsoid of inertia about the fixed point is not an ellipsoid of revolution then with the exception of the case $x_0 = y_0 = z_0 = 0$, an additional algebraic integral of the system (1.1) cannot exist.

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